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NON-ABELIAN VORTEX AND CONFINEMENT

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Abstract: We discuss general properties and possible types of magnetic vortices in non-Abelian gauge theories (we consider here $G = SU(N), SO(N), USp(2N)$) in the Higgs phase. The sources of such vortices carry “fractional” quantum numbers such as Z_n charge (for $SU(N)$), but also full non-Abelian charges of the dual gauge group. If such a model is re-interpreted as an effective dual magnetic theory of the fundamental (electric) theory, the non-Abelian vortices provide for the mechanism of quark-confinement in the latter. We compare this mechanism with the dynamical Abelian mechanism for QCD suggested by ’t Hooft and Mandelstam.

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1. Introduction

The mechanism of confinement and dynamical symmetry breaking, and the relation thereof, has recently been studied in detail, in a class of asymptotically free $\mathcal{N} = 2$ supersymmetric gauge theories (in which supersymmetry is softly broken to $\mathcal{N} = 1$) with various gauge groups, $SU(n_c)$, $USp(2n_c)$ and $SO(n_c)$, and with different numbers of flavors [1], generalizing the pioneering works of Seiberg and Witten [2, 3]. These models are characterized by the existence of a large (discrete) vacuum degeneracy, even in the presence of nonvanishing, generic matter masses, and as a result a theory with a given Lagrangean can and does in fact display distinct vacua in various phases.

An advantage of studying such theories lies in the fact that the low-energy degrees of freedom and the form of the effective action can be determined explicitly, by combining duality, supersymmetry, Seiberg-Witten exact curves [4] and some knowledge on superconformal theories [5]. Thus we learn that, even if we restrict ourselves to vacua in confinement phase, there are distinct types of confining phases, distinguished by different entities which condense. In some vacua they are monopoles of the maximal Abelian subgroup of the gauge group G , as envisaged by 't Hooft and Mandelstam [6]. More typically, however, they are magnetic monopoles (dual quarks) carrying non-Abelian charges, and interacting with light gauge bosons of a non-Abelian effective gauge theory. It is possible also, as in an important class of vacua in $USp(2n_c)$ and $SO(n_c)$ theories, confinement is due to the cooperation of relatively non-local dyons. The effective theory is a deformation of a nontrivial superconformal theory in these vacua.

Also, the question of which fields appear as the low-energy effective degrees of freedom, was found to be intimately related to the pattern of dynamical symmetry breaking [1].

Another interesting model is the $SU(N)$ gauge theory with $N = 4$ supersymmetry, in which supersymmetry is softly broken to $N = 1$ by the masses of the three adjoint scalar multiplets. Dual Meissner effect occurs in the confining vacuum, but without dynamical Abelianization [7].

In the case of the standard, nonsupersymmetric QCD, where there is a unique vacuum, if confinement occurs the system must choose a particular type of confinement mechanism. Which among the above mentioned possibilities is realized in QCD

is a dynamical question. Certain problems in the Abelian mechanism of confinement for QCD, have been pointed out [8, 7, 9].

Inspired by these new developments, we propose here to study certain general properties of vortices appearing in a non-Abelian gauge theory, in which the continuous gauge symmetry is completely broken by the Higgs mechanism, leaving a discrete center unbroken. These vortex solutions are generalizations of Nielsen-Olesen-Abrikosov vortices of $U(1)$ gauge theory [10, 11], but are associated with “fractional” quantum numbers, such as \mathbb{Z}_N quantum number (N -ality), for instance, in the case of $SU(N)$ theory. Quite detailed studies of explicit vortex-like solutions in the cases of $SU(N)$ theories have been made earlier by de Vega and Schaposnik [12, 14] and more recently by several authors [15]-[17], by using some explicit forms of scalar potentials. See also [13].

Here we shall concentrate on more general properties of the vortices and quantum numbers carried by their possible sources. One of the most interesting findings below is that the sources of such vortices in a given theory G , carry also full quantum number of the non-Abelian dual group \tilde{G} . For instance, $\tilde{G} = SU(N)$ for $G = SU(N)/\mathbb{Z}_N$. Sources of the gauge invariant set of the vortices precisely correspond to irreducible multiplets of \tilde{G} , represented by the Young tableaux with $n = 1, 2, \dots, N-1$ boxes.

Which of these vortices other than the lowest, N -ality one vortex, represent stable vortices (and not just bundles of lower vortices), and what the relative tensions among them are, etc., are questions depending on the details of dynamics, quantum effects, etc., going beyond the scope of the general discussions in this paper.

2. An illustration: $SO(3) = SU(2)/\mathbb{Z}_2$ Vortex

In the case of a $SO(3) = SU(2)/\mathbb{Z}_2$ model, the Lagrangean is

$$L = -\frac{1}{4}\hat{F}_{\mu\nu}^2 + \frac{1}{2}[(D_\mu\hat{\phi}_1)^2 + (D_\mu\hat{\phi}_2)^2] - V(\hat{\phi}_1, \hat{\phi}_2), \quad (2.1)$$

where

$$\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu + g\hat{A}_\mu \times \hat{A}_\nu; \quad D_\mu\hat{\phi}_i = (\partial_\mu + g\hat{A}_\mu \times)\hat{\phi}_i; \quad (2.2)$$

and the potential can be taken, for instance, as

$$V(\hat{\phi}_1, \hat{\phi}_2) = \sum_{A=1}^2 \lambda_A (\hat{\phi}_A \cdot \hat{\phi}_A - F_A^2)^2 + \kappa (\hat{\phi}_1 \cdot \hat{\phi}_2 - G^2)^2. \quad (2.3)$$

The potential is taken such that the gauge group $SO(3)$ is broken spontaneously at its minima. The minimum of the potential is at

$$\hat{\phi}_A \cdot \hat{\phi}_A = F_A^2 \quad (A = 1, 2); \quad \hat{\phi}_1 \cdot \hat{\phi}_2 = G^2 \equiv F_1 F_2 \cos \Theta, \quad (2.4)$$

where we have assumed $|\frac{G^2}{F_1 F_2}| < 1$. By a gauge transformation $\hat{\phi}_1$ can be taken in the 3 direction (of the isospin space). Existence of regular solutions of vortex type can be easily worked out, which have the asymptotic behavior at large r (where we introduce the cylindrical coordinates (r, ϕ, z))

$$\hat{\phi}_1 \sim U_n(\phi) \begin{pmatrix} F_1 \\ 0 \\ 0 \end{pmatrix}; \quad \hat{\phi}_2 \sim U_n(\phi) U_1(\Theta) \begin{pmatrix} F_2 \\ 0 \\ 0 \end{pmatrix}; \quad A_i \sim U_n(\phi) \frac{1}{g} \partial_i U_n(\phi)^\dagger, \quad (2.5)$$

with

$$U_n(\phi) = \begin{pmatrix} \cos n\phi & -\sin n\phi & 0 \\ \sin n\phi & \cos n\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.6)$$

As this analysis overlaps substantially with the results of earlier works [12]-[16], we shall discuss it only briefly in Appendix A. Note that \hat{A}_i and \hat{A}_{ij} are nonvanishing only for $i = 1, 2$. As one goes around the string (which is taken to be along the z axis), the vector $\hat{\phi}_1, \hat{\phi}_2$ rotate n times around the 3 axis of the $SO(3)$ space: it gives rise apparently to a vortex with winding number n .

Indeed, following [11] one can show that for a static vortex,

$$\int_{S_1} \hat{F}_{ij} d\sigma_{ij} - \int_{S_2} \hat{F}_{ij} d\sigma_{ij} = 0, \quad i, j = 1, 2 \quad (2.7)$$

where $S_{1,2}$ are the bottom and top circles of the cylindrical region considered, apparently allowing one to define a “conserved” flux. For the above mentioned string configuration, one finds

$$\int_S F_{ij}^3 d\sigma_{ij} = \oint dx_i A_i^3 = \frac{2\pi n}{g}, \quad (2.8)$$

where the circulation is taken at a large radius r .

The problem with this definition of the flux is that it is not gauge invariant. In fact, in the case of the vortex with flux two ($n = 2$), it is possible to construct explicitly (Appendix B) a gauge transformation, regular everywhere, such that

$$U_{global}(\phi, r) \xrightarrow{r \rightarrow \infty} U_2(-\phi); \quad U_{global}(\phi, r) \xrightarrow{r \rightarrow 0} \mathbf{1}. \quad (2.9)$$

By using such a gauge transformation, the apparent vortex can be gauged away. This corresponds to the well-known fact the 4π rotation in the $SO(3)$ space can be smoothly shrunk to a point. On the other hand, for the minimum winding number $n = 1$, this is not possible due to the fact that

$$\Pi_1(SO(3)) = \Pi_1(SU(2)/\mathbb{Z}_2) = \mathbb{Z}_2. \quad (2.10)$$

Sources of the vortices in $SO(3)$ theory carry thus the unique \mathbb{Z}_2 charge.¹

An analogous procedure might appear to be able to eliminate the winding of the gauge fields hence the flux analogous to (2.8) in the Abelian, Abrikosov-Nielsen-Olesen vortices. The crucial difference is that in the latter case any such gauge transformation necessarily introduces a singularity in the gauge fields along the core of the vortex ($r = 0$): the flux is now concentrated within an infinitely thin vortex core: vortex cannot be eliminated in an Abelian case, whatever its winding number may be. This fact reflects the topological property

$$\Pi_1(U(1)) = \mathbb{Z}. \quad (2.11)$$

3. General characterization of vortices: flux quantization

The general properties of the vortex - possible types of vortices - are independent of the detailed form of the scalar potential or of the number of the Higgs fields present: they are determined by the asymptotic behavior of the fields. The latter should be such that far from the vortex the gauge fields are pure gauge form, and the matter scalar fields are covariantly constant and at the minimum of the potential. With an appropriate gauge choice such fields can be taken, far from the core of the vortex (which we take along the z axis), as

$$A_i \sim \frac{i}{g} U(\phi) \partial_i U^\dagger(\phi); \quad \phi_A \sim U \phi_A^{(0)} U^\dagger, \quad U(\phi) = \exp i \sum_j^r \beta_j T_j \phi \quad (3.1)$$

where $\phi_A^{(0)}$ are fixed scalar VEVs at a minimum of the potential, such that the gauge group G is completely broken, except a discrete center, and T_i 's are the generators of

¹For the same mathematical reason, the Dirac like magnetic monopoles in an *unbroken* $SO(3)$ theory carry the unique \mathbb{Z}_2 charge [18].

the Cartan subalgebra of G . Since T_i 's commute with each other, one has

$$A_\phi \sim \frac{1}{g r} \sum_j^r \beta_j T_j \quad (3.2)$$

so that the above vortex carry the flux

$$\oint dx_i A_i = \frac{2\pi}{g} \sum_j^r \beta_j T_j. \quad (3.3)$$

Such a flux is well-defined modulo gauge transformation which permute T_i 's (Weyl transformations), hence β_i 's:²

$$S_\alpha \left(\sum_j^r \beta_j T_j \right) S_\alpha = \sum_j^r \beta'_j T_j, \quad \beta' = \beta - \frac{2\alpha(\beta \cdot \alpha)}{(\alpha \cdot \alpha)}, \quad (3.4)$$

$$S_\alpha = \exp[i\pi(E_\alpha + E_{-\alpha})/\sqrt{2\alpha^2}], \quad (3.5)$$

where α is a root vector and E_α is a nondiagonal generator in the Cartan basis.

The quantization condition on β_j follows from the requirement that the fields are single valued, i.e.,

$$U(2\pi) \in \mathcal{C}, \quad (3.6)$$

where \mathcal{C} is the center of the group G . For $SU(N)$, for instance, $\mathcal{C} = \mathbb{Z}_N$. For $SO(2N)$, $N \geq 2$, $\mathcal{C} = \mathbb{Z}_2$. The fact that $U(2\pi) \in \mathcal{C}$ commutes with all generators leads, by commuting it with E_α 's, to the general condition for β ,³

$$\alpha \cdot \beta \in \mathbb{Z}. \quad (3.7)$$

Thus the minimum vortex of $SU(N)$, for instance, corresponds to $U_{min}(\phi)$ with

$$U_{min}(2\pi) = e^{2\pi i/N} \mathbf{1}, \quad (3.8)$$

which represents a noncontractible loop of the coset space $SU(N)/\mathbb{Z}_N$. On the other hand, a bundle of N vortices or a single vortex with N windings

$$U(\phi) = \exp iN \sum_j^r \beta_j T_j \phi = [U_{min}(\phi)]^N \quad (3.9)$$

²We follow the notation of Goddard, Nuyts and Olive [19].

³Note that our β are twice as large as compared to β appearing in the problem of nonAbelian monopoles [19].

satisfies

$$U(2\pi) = \mathbf{1} : \quad (3.10)$$

it is a loop in $SU(N)$. Since $SU(N)$ is simply connected, such a loop can be smoothly contracted to a point, as r is reduced from $r = \infty$ to $r = 0$, as in the example of $SU(2)/\mathbb{Z}_2$ theory discussed in the previous section.

The general condition Eq.(3.7) implies that the “charges” β characterizing the vortex live in the dual group \tilde{G} of G , as shown by Goddard, Nuyts and Olive [19] in an analogous problem with the monopoles. This follows from the fact that the root vectors of the group in which the charges β live form the dual lattice of the original root lattice. The examples of such dual groups are:

$$\begin{aligned} SU(N) &\leftrightarrow SU(N)/\mathbb{Z}_N; \\ SO(2N) &\leftrightarrow SO(2N); \\ SO(2N+1) &\leftrightarrow USp(2N). \end{aligned} \quad (3.11)$$

Physical settings studied by Goddard, Nuyts and Olive [19] (monopoles) are however different from the ones studied here (vortices). Regular monopole solutions appear in a theory in which the gauge group G is broken spontaneously to an unbroken subgroup H by Higgs mechanism,

$$G \xrightarrow{\langle \Phi \rangle \neq 0} H. \quad (3.12)$$

such that

$$\Pi_2(G/H) \neq \mathbf{1}. \quad (3.13)$$

($\Pi_2(G/H) = \Pi_1(H)$ if G is simply connected). The first example was found originally in the theory $G = SU(2)$; $H = U(1)$ [20]. Their possible charges, with respect to the unbroken group H , are characterized [19] by the weight vectors of the dual group \tilde{H} .

The vortices studied here appear in theories in which the gauge group G is completely broken by the Higgs mechanism leaving some unbroken discrete center, such that the fundamental group,

$$\Pi_1(G/C) = C, \quad (3.14)$$

(if G is simply connected) is nontrivial. The sources of these gauge invariant set of vortices, carry quantum numbers correspond to the irreducible representations of the dual group \tilde{G} (of G), as will be shown below.

4. Solutions of the quantization condition; quantum numbers of the sources

The quantization condition on the vortices Eq.(3.6), Eq.(3.7) will be solved now explicitly, and the solutions will be classified, for $SU(3)$, $SU(N)$, $SO(2N)$, $SO(2N+1)$ and $USp(2N)$ gauge groups.

4.1. $SU(3)$

By normalizing the generators in a canonical way, we have

$$T_3 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad T_8 = \frac{1}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (4.1)$$

The normalization of the generators is such that the metric of the root vector space satisfies

$$g^{ij} = \sum_{\text{roots}} \alpha^i \alpha^j = \delta^{ij}. \quad (4.2)$$

With this normalization the asymptotic gauge transformation matrix becomes

$$U(\phi) = \exp i \sum_j^r \beta_j T_j \phi \longrightarrow \begin{pmatrix} e^{i\phi(\beta_3/2\sqrt{3}+\beta_8/6)} & 0 & 0 \\ 0 & e^{i\phi(-\beta_3/2\sqrt{3}+\beta_8/6)} & 0 \\ 0 & 0 & e^{-i\phi\beta_8/3} \end{pmatrix}. \quad (4.3)$$

The quantization condition $U(2\pi) \in \mathbb{Z}_3$ gives

$$\frac{\beta_3}{2\sqrt{3}} + \frac{\beta_8}{6} = -\frac{n_1}{3}, \quad -\frac{\beta_3}{2\sqrt{3}} + \frac{\beta_8}{6} = -\frac{n_2}{3}, \quad -\frac{1}{3}\beta_8 = -\frac{n_3}{3}, \quad (4.4)$$

where for a vortex of minimum winding,

$$n_i = [1 \bmod 3], \quad \sum_i n_i = 0. \quad (4.5)$$

The solution for $\beta = (\beta_3, \beta_8)$ is in this case

$$\beta = (-\sqrt{3}, 1), (\sqrt{3}, 1), \text{ or } (0, -2), \quad (4.6)$$

namely,

$$\beta = 6 \mathbf{w} = 2N \mathbf{w}, \quad (4.7)$$

where \mathbf{w} is a weight vector of the fundamental representation, $\underline{3}$. Obviously, if one chooses the weight vector of the anti-fundamental representation,

$$\beta = 6 \bar{\mathbf{w}} = -6 \mathbf{w}, \quad (4.8)$$

one gets instead a vortex with flux minus one.

Obviously, $U'(\phi) = U(\phi)^3$, $\phi : 0 \rightarrow 2\pi$ is a closed loop in $SU(3)$, hence such a vortex can be gauged away.

Various choices for β in Eq.(4.7) are related by Weyl transformations,

$$\mathbf{w} \rightarrow \mathbf{w} - \frac{2\alpha(\mathbf{w} \cdot \alpha)}{(\alpha \cdot \alpha)}, \quad (4.9)$$

which can be obtained by a gauge transformation of the form, Eq.(3.4). In other words, a vortex with one solution of Eq.(4.4), Eq.(4.5) and another vortex related to it by a Weyl transformation, are gauge equivalent. These correspond precisely to the possible sources of such magnetic vortex which carry the quantum numbers of the representation $\underline{3}$, and a unit triality. We have a unique gauge-invariant \mathbb{Z}_3 vortex of triality one.

The solutions with triality two can found by choosing the solution of

$$\frac{\beta_3}{2\sqrt{3}} + \frac{\beta_8}{6} = -\frac{n_1}{3}, \quad -\frac{\beta_3}{2\sqrt{3}} + \frac{\beta_8}{6} = -\frac{n_2}{3}, \quad -\frac{1}{3}\beta_8 = -\frac{n_3}{3}, \quad (4.10)$$

with

$$n_i = [2 \bmod 3], \quad \sum_i n_i = 0. \quad (4.11)$$

From the point of view of \mathbb{Z}_3 quantum numbers, of course, the triality two is equivalent to minus one; however a priori this may not be the whole story. The source of these vortices can carry also the full $SU(3)$ quantum numbers. In the case of flux two, there are in fact two solutions,

$$\beta = (-\sqrt{3}, -1), (\sqrt{3}, -1), \text{ or } (0, 2), \quad (4.12)$$

which are the weight vectors of $\underline{3}^*$, and

$$\beta = (-2\sqrt{3}, 2), (2\sqrt{3}, 2), (0, 2), (-\sqrt{3}, -1), (\sqrt{3}, -1), \text{ or } (0, -4), \quad (4.13)$$

which are the weight vectors of $\underline{6}$.

Classically these correspond to two distinct gauge-invariant sets of vortices. However, quantum mechanically, the vortex with the higher tension (probably $\underline{6}$) will decay into the one with the lower tension (probably $\underline{3}^*$) through the gauge boson pair productions (Fig. 1). We thus find a unique \mathbb{Z}_3 vortex in the $SU(3)$ gauge theory.

As the dual of $SU(3)/\mathbb{Z}_3$ is precisely $SU(3)$, the vortex found above corresponds to the confining string for the quarks if the present model is re-intepreted as the magnetic $SU(3)$ theory (see Sec.5. below).

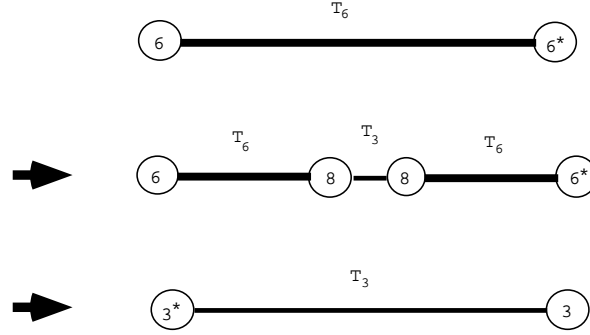


Figure 1: A vortex with a higher tension decays into one with a lower tension through the pair production of particles in the adjoint representation.

4.2. $SU(N)$ gauge theory

The diagonal generators can be taken as

$$\mathbf{T}_i = \begin{pmatrix} w_1^i & & & & \\ & w_2^i & & & \\ & 0 & \ddots & & \\ & & & w_{N-1}^i & \\ & & & & w_N^i \end{pmatrix}, \quad i = 1, 2, \dots, N-1 \quad (4.14)$$

where w_k represents the k -th weight vector of the fundamental representation of $SU(N)$, satisfying [21]

$$\mathbf{w}_k \cdot \mathbf{w}_l = -\frac{1}{2N^2}; \quad (k \neq l); \quad \mathbf{w}_k \cdot \mathbf{w}_k = \frac{N-1}{2N^2}, \quad k, l = 1, 2, \dots, N. \quad (4.15)$$

They are vectors in $N - 1$ dimensional Euclidean space. The quantization condition for a vortex is

$$U(2\pi) \in \mathcal{C} = \mathbb{Z}_N, \quad U(\phi) = \exp i \sum_j^r \beta_j T_j \phi \quad (4.16)$$

which reads for the minimum flux:

$$\sum_{i=1}^{N-1} \beta^i w_k^i = \beta \cdot \mathbf{w}_k = -\frac{n_k}{N}, \quad k = 1, 2, \dots, N, \quad (4.17)$$

with

$$n_k = \{1 \bmod N\}, \quad \sum_{k=1}^N n_k = 0. \quad (4.18)$$

To find the solutions to these equations we first note that of N equations (4.17) only $N - 1$ are independent, due to the fact that

$$\sum_{k=1}^N \mathbf{w}_k^i = 0. \quad (4.19)$$

One particular solution of N equations (4.17),(4.18) can then be found by e.g. choosing

$$n_2 = n_3 = \dots = n_N = 1, \quad n_1 = -(N - 1) \quad (4.20)$$

With this choice, the vector $\tilde{\beta} = \beta/2N$ forms the same scalar products with $N - 1$ weight vectors $\mathbf{w}_2, \mathbf{w}_3, \dots, \mathbf{w}_N$, as does \mathbf{w}_1 . (See Eq.(4.16).) This proves that

$$\tilde{\beta} = \mathbf{w}_1 \quad \therefore \quad \beta = 2N\mathbf{w}_1 \equiv \beta_1, \quad (4.21)$$

for if $\tilde{\beta} - \mathbf{w}_1$ were not a null vector, it would be orthogonal to each vector in the complete set, $\{ \mathbf{w}_2, \mathbf{w}_3, \dots, \mathbf{w}_N \}$, which is an absurdity.

Other solutions of (4.17), (4.18) can be found by choosing different set of $N - 1$ n_i 's to be equal to 1, and the remaining one (say, n_j) to be equal to $-(N - 1)$. The corresponding vortex has

$$\beta_j = 2N\mathbf{w}_j, \quad j = 2, 3, \dots, N. \quad (4.22)$$

We thus find that the possible vortices of minimum flux are characterized precisely by the weight vectors of the fundamental representation of $SU(N)$, generalizing the result found for $SU(3)$.

Note that these N solutions are not independent: they are related by gauge transformations hence physically equivalent. There is thus a unique vortex of N -ality one.

The vortices of the minimum flux minus one (N -ality $= -1$) can be found analogously, with a plus sign on the right hand side in E.(4.17), hence by changing the sign of β_j in the solutions found above.

There are other solutions of (4.17), (4.18) representing vortices of higher N -alities. At the N -ality two, for instance, the solutions for β have the form,

$$2N(\mathbf{w}_i + \mathbf{w}_j), \quad i, j = 1, 2, \dots, N. \quad (4.23)$$

They fall into two gauge inequivalent sets of vortices: their sources would carry the quantum numbers of the two irreducible representations,

$$\square\square, \quad \begin{smallmatrix} \square \\ \square \end{smallmatrix}, \quad (4.24)$$

symmetric and antisymmetric in color, respectively.

Solutions of N -ality k can be analogously be constructed by taking as β the vector sum of arbitrary k minimum solutions, Eq.(4.22). These vortices can be grouped into gauge invariant subsets, each of which has a source carrying quantum numbers of an irreducible representations of $SU(N)$ group,

$$\overbrace{\square\square\dots\square\square}^k, \quad \overbrace{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \dots \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}^{k-1}, \quad \dots, \quad \begin{smallmatrix} \square \\ \square \\ \square \\ \square \end{smallmatrix}, \quad (4.25)$$

all having k boxes.

The vortices of N -ality, $1, 2, \dots, N - 1$ cannot be unwound by a gauge transformation, representing elements of the fundamental group

$$\Pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N. \quad (4.26)$$

Nevertheless, this does not mean that each of the vortices (4.25) is stable against decay. A vortex of a given N -ality can decay through the pair production of gauge bosons into one of the same \mathbb{Z}_N quantum number but with a lower tension, via processes similar to the one in the $SU(3)$ example of Fig. 1. It is possible that the tension is smallest in the case of the antisymmetric representation $\underline{\binom{N}{k}}$. In other words,

the solution for the vortex charge β at N -ality k is truly a unique gauge-invariant set

$$2N \{ \mathbf{w}_{i_1} + \mathbf{w}_{i_2} + \dots + \mathbf{w}_{i_k} \mod \alpha \}, \quad i_m = 1, 2, \dots, N, \quad (4.27)$$

where α 's are the root vectors of the $SU(N)$ group.

Which of these, apart from the smallest, N -ality one vortex, is stable against decay into a bundle of vortices with smaller N -alities, is a dynamical question (i.e., depends on the form of the potential, values of coupling constants, quantum corrections, etc.). One would expect no universal formula for the relative tensions among vortices of different N -alities on the general ground. However, there are some intriguing suggestions [22] that the ratios among the vortex tensions for different \mathbb{Z}_N charges, found originally in the pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory (broken softly to $\mathcal{N} = 1$) [8],

$$T_k \propto \sin \frac{\pi k}{N}, \quad (4.28)$$

might be universal, and equally interesting results from lattice simulation [23, 24], concerning these issues.

The absence of vortices of N -ality, N , can be understood since the charges corresponding to an irreducible representation with N boxes in the Young tableau, can always be screened by those of the dynamical fields (adjoint representation): the vortex is broken by copious production of massless gluons of the dual $SU(N)$ theory.

Finally, it is easy to prove that these solutions do satisfy the general condition Eq.(3.7). In fact, for each root vector α , one finds from Eq.(4.22)

$$\alpha \cdot \beta_j = 2N \alpha \cdot \mathbf{w}_j = N(\alpha \cdot \alpha) \times \text{integer}, \quad (4.29)$$

where we have used the well known theorem stating that for any root vector α and for any weight vector \mathbf{w} , $2(\alpha \cdot \mathbf{w})/(\alpha \cdot \alpha)$ is an integer. For $SU(N)$, $\alpha \cdot \alpha = 1/N$, thus the general condition Eq.(3.7) is indeed met by our solutions.

4.3. $SO(2N)$ gauge theory

The generators in the Cartan subalgebra of $SO(2N)$ group can be taken as

$$T_i = \begin{pmatrix} -iw_1^i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} & & & \\ & -iw_2^i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} & & \\ & & \ddots & \\ & & & -iw_N^i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \end{pmatrix}, \quad (4.30)$$

($i = 1, 2, \dots, N$) where \mathbf{w}_k ($k = 1, 2, \dots, N$) are the weight vectors of the fundamental representation, living in an N -dimensional Euclidean space and satisfying

$$\mathbf{w}_k \cdot \mathbf{w}_l = 0; \quad k \neq l; \quad \mathbf{w}_k \cdot \mathbf{w}_k = \frac{1}{4(N-1)} : \quad (4.31)$$

they form a complete set of orthogonal vectors.

The form of the minimal vortex depends on the field content of the theory. If the scalar fields involved are all assumed to transform as tensor representations of even ranks (for instance, the antisymmetric representation), then the gauge group is effectively $SO(2N)/\mathbb{Z}_2$, with

$$\Pi_1(SO(2N)/\mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2. \quad (4.32)$$

The quantization condition for a minimum nontrivial $SO(2N)$ vortex is that $U(\phi)$ appearing in the asymptotic behavior of the fields Eq.(3.1) behaves in this case as

$$U(\phi) = \exp i \phi \sum_{j=1}^N \beta^j T_j \xrightarrow{\phi=2\pi} \begin{pmatrix} -\mathbf{1}_{2 \times 2} & & & \\ & -\mathbf{1}_{2 \times 2} & & \\ & & \ddots & \\ & & & -\mathbf{1}_{2 \times 2} \end{pmatrix} = -\mathbf{1}_{2N \times 2N}. \quad (4.33)$$

It means that β has a general form,

$$\beta = 2(N-1) \{ \pm \mathbf{w}_1 \pm \mathbf{w}_2 \dots \pm \mathbf{w}_N \}, \quad (4.34)$$

so that

$$\beta \cdot \mathbf{w}_i = \pm \frac{1}{2}, \quad i = 1, 2, \dots, N. \quad (4.35)$$

To see the consistency with the general condition, Eq.(3.7), we note that the root vectors of $SO(2N)$ group are $\alpha = \pm \mathbf{w}_i \pm \mathbf{w}_j$ ($i \neq j$). One finds then

$$\beta \cdot \alpha = \pm 1, 0 \quad (4.36)$$

for any solution of the form (4.34).

There are 2^N solutions with the minimum flux, (4.34). Half of them (2^{N-1}) contain even number of minus signs, the other half an odd number of minus signs. Note that the Weyl transformations (Eq.(3.4)) transform among these solutions by permutations, leaving however the set with even (or odd) number of minus signs invariant.⁴ Thus the 2^{N-1} solutions with even number of minus signs form an irreducible representation. So do 2^{N-1} solutions with odd numbers of minus signs, in another irreducible representation. These are precisely the multiplicities of chirality ± 1 spinor representations of the $SO(2N)$ group. *We conclude that the sources of the vortices in $SO(2N)/\mathbb{Z}_2$ theory carry the quantum numbers of the chirality ± 1 spinor representations of the (dual) $SO(2N)$ group.*

Consider instead a theory in which scalar fields in the vector (or any odd-rank tensor) representation get vacuum expectation values. The gauge group is now truly $SO(2N)$, with

$$\Pi_1(SO(2N)) = \mathbb{Z}_2. \quad (4.37)$$

The vortex representing the nontrivial element of this \mathbb{Z}_2 is of the form, Eq.(3.1), with

$$U(\phi) = \exp i \phi \sum_{j=1}^N \beta^j T_j \xrightarrow{\phi=2\pi} \begin{pmatrix} \mathbf{1}_{2 \times 2} & & & \\ & \mathbf{1}_{2 \times 2} & & \\ & & \ddots & \\ & & & \mathbf{1}_{2 \times 2} \end{pmatrix} = \mathbf{1}_{2N \times 2N} : \quad (4.38)$$

The minimal solutions for β_i are then:

$$\beta = \pm 4(N-1) \mathbf{w}_i, \quad i = 1, 2, \dots, N : \quad (4.39)$$

corresponding to the sources in the vector ($2N$) representation on the (dual) $SO(2N)$ theory.

⁴For instance a π rotation in the $(1-2)$ plane changes $\mathbf{w}_1 \rightarrow -\mathbf{w}_1$, $\mathbf{w}_2 \rightarrow -\mathbf{w}_2$, while leaving other \mathbf{w}_i 's untouched.

4.4. $SO(2N + 1)$ gauge theory

The N generators in the Cartan subalgebra of $SO(2N + 1)$ group can be taken as

$$T_i = \begin{pmatrix} -iw_1^i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} & & & \\ & -iw_2^i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} & & \\ & & \ddots & \\ & & & -iw_N^i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \\ & & & & 0 \end{pmatrix}, \quad (4.40)$$

where \mathbf{w}_k ($k = 1, 2, \dots, N$) are the weight vectors of the fundamental representation, living in an N -dimensional Euclidean space and satisfying

$$\mathbf{w}_k \cdot \mathbf{w}_l = 0; \quad k \neq l; \quad \mathbf{w}_k \cdot \mathbf{w}_k = \frac{1}{2(2N - 1)} : \quad (4.41)$$

they form a complete set of orthogonal vectors. The quantization condition in this case is:

$$U(2\pi) = \exp i 2\pi \sum_{j=1}^N \beta^j T_j = \begin{pmatrix} \mathbf{1}_{2 \times 2} & & & \\ & \mathbf{1}_{2 \times 2} & & \\ & & \ddots & \\ & & & \mathbf{1}_{2 \times 2} \\ & & & & 1 \end{pmatrix} = \mathbf{1}_{(2N+1) \times (2N+1)}. \quad (4.42)$$

The minimal solutions for β_i are then $2N$ solutions:

$$\beta = \pm 2(2N - 1) \mathbf{w}_i, \quad i = 1, 2, \dots, N, \quad (4.43)$$

which can be regarded as the weight vectors of the fundamental representation of $USp(2N)$ group which is dual to $SO(2N + 1)$ (see Eq.(3.11)).

As the root vectors of $SO(2N + 1)$ group are $\alpha = \{\pm \mathbf{w}_i, \pm \mathbf{w}_i \pm \mathbf{w}_j\}$, the general condition Eq.(3.7) is satisfied minimally (i.e., $\beta \cdot \alpha = \pm 1$).

On the other hand, nonminimal solutions obtained by combining two minimal solutions β_1 and β_2 of the form, (4.43), make up a set with

$$\frac{\beta}{2(2N - 1)} = \pm 2\mathbf{w}_i, \pm \mathbf{w}_i \pm \mathbf{w}_j : \quad (4.44)$$

these correspond (apart from an overall normalization) precisely to the root vectors of the dual group $USp(2N)$. These vortices can be gauge-transformed away as $\Pi_1(SO(2N + 1)) = \mathbb{Z}_2$.

4.5. $USp(2N)$ gauge theory

The N generators in the Cartan subalgebra of $USp(2N)$ group are the following $2N \times 2N$ matrices,

$$\mathbf{T}_i = \begin{pmatrix} \mathbf{B}_i & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_i^t \end{pmatrix}, \quad i = 1, 2, \dots, N, \quad (4.45)$$

where

$$\mathbf{B}_i = \begin{pmatrix} w_1^i & & & \\ & w_2^i & & \\ & 0 & \ddots & 0 \\ & & & w_{N-1}^i \\ & & & & w_N^i \end{pmatrix}, \quad i = 1, 2, \dots, N. \quad (4.46)$$

The weight vectors \mathbf{w}_k ($k = 1, 2, \dots, N$) form a complete set of orthogonal vectors in an N -dimensional Euclidean space and satisfy

$$\mathbf{w}_k \cdot \mathbf{w}_l = 0; \quad k \neq l; \quad \mathbf{w}_k \cdot \mathbf{w}_k = \frac{1}{4(N+1)}. \quad (4.47)$$

The quantization condition for a vortex is $(\Pi_1(USp(2N)/\mathbb{Z}_2) = \mathbb{Z}_2)$

$$U(2\pi) = -\mathbf{1}_{2N \times 2N}, \quad U(\phi) = \exp i \sum_j^r \beta_j \mathbf{T}_j \phi \quad (4.48)$$

which reads for the minimum flux:

$$\sum_{i=1}^N \beta^i w_k^i = \beta \cdot \mathbf{w}_k = \pm \frac{1}{2}, \quad k = 1, 2, \dots, N. \quad (4.49)$$

The minimum solutions are

$$\beta = 2(N+1) \{ \pm \mathbf{w}_1 \pm \mathbf{w}_2 \dots \pm \mathbf{w}_N \}. \quad (4.50)$$

These can be interpreted as the weight vectors of the 2^N dimensional spinor representation of the dual group, $SO(2N+1)$.⁵

Also in this case the general condition Eq.(3.7) is easily seen to be fulfilled as the root vectors of $USp(2N)$ are given by (4.44).

⁵In contrast to the case of the $SO(2N)$ theory, the signs in Eq.(4.50) can be changed singly by a gauge transformation. For instance, to change the sign of \mathbf{w}_1 , consider an $SU(2)$ subgroup of $USp(2N)$ acting in the $i, j = 1, N+1$ subspace and make a π rotation with $\sigma_1/2$.

5. Non-Abelian duality and confinement in $SU(N)$ theories

Classification of the possible phases of $SU(N)$ gauge theory at zero temperature was discussed by 't Hooft, by making crucial use of electromagnetic duality [25, 26]. The fundamental role is played by the quantum numbers of the entity which condenses. If magnetically charged particles condense (magnetic Higgs phase), the theory is in a confinement phase, the Wilson loop displaying the area law. If electrically charged particles get VEVs, the theory is in a Higgs phase. If no particles condense, the theory is in a Coulomb phase, with a long range force between charged particles. Intermediate phases where particles charged both electrically and magnetically (dyons) condense, are also possible. In the pure $SU(N)$ Yang Mills theory, these charges - the external electric charges which can be introduced as a probe and which cannot be screened by the dynamical fields, and the possible values of the magnetic charges - are both classified by the center charge of $SU(N)$, \mathbb{Z}_N . If a field with $(\mathbb{Z}_N, \mathbb{Z}_N) = (a, b)$ condenses, any particles (c, d) having a nonvanishing relative Dirac unit with respect to it, $ad - bc \neq \{0 \bmod N\}$, are confined, while those having $ad - bc = \{0 \bmod N\}$ are not.

However, such a classification is not quite complete. Also, it was not clear in such an approach which the full set of effective low-energy degrees of freedom were and how the low-energy effective Lagrangean looked like.

Subsequently a more physical picture of confinement was proposed by 't Hooft through the so-called Abelian gauge fixing [6]. *Assuming* that the relevant degrees of freedom are the monopoles that appear as singularities of such a gauge fixing, the confinement is understood as the dual Meissner effect of the effective $U(1)^2 \in SU(3)$ theory. This proposal was widely studied in the literature; however, the correctness of the hypothesis of the dynamical Abelianization remains an open question (for instance, see [27]).

Recently, however, problems with the Abelian mechanism of confinement (as a model for QCD) have been pointed out [8, 9]. A crucial aspect of this mechanism is that $SU(3)$ gauge group is dynamically broken to the Abelian subgroup, $U(1)^2$. Higgs mechanism in the dual $U(1)^2$ theory leads to confinement of the original theory. Dynamical $SU(3) \rightarrow U(1)^2$ breaking however in general implies the existence of more than one meson Regge trajectories. More precisely, these Regge trajectories are not linear; all of them however approach the universal, linear trajectory at large masses

[8, 9]. This is certainly not the pattern observed in Nature. Dynamical Abelianization does nonetheless occur in certain supersymmetric models [2, 3, 8, 1].

Consider instead the $N = 4$ supersymmetric YM theory in which supersymmetry is broken softly to $N = 1$ by small adjoint scalar masses. Dual Meissner effect takes place in a confining vacuum, but the duality involved is of Olive-Montonen type [28], $SU(N) \leftrightarrow SU(N)/\mathbb{Z}_N$. Such a model predicts a unique, universal $q - \bar{q}$ meson Regge trajectory, and may be a better model of confinement in QCD (and hopefully, in the same universality class) [7].

Our analysis of previous sections show that the vortices appearing in the Higgs phase of the dual, magnetic $SU(N)/\mathbb{Z}_N$ theory (in the case of the original, $SU(N)$ Yang-Mills theory) have exactly the right properties required to confine quarks, if these are introduced in the electric theory.

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Appendix A: Explicit solution for $SO(3)$ vortex

Existence of a regular vortex like solution in the $SO(3) = SU(2)/\mathbb{Z}_2$ theory discussed in Sec. 2. can be shown by setting

$$A_\phi^3 = A_\phi^3(r) \neq 0, \quad A_\mu^a = 0, \quad \text{otherwise}, \quad (\text{A.1})$$

$$\hat{\phi}_1 = \psi_1(r) U_n(\phi) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \psi_1(r) \begin{pmatrix} \cos n\phi \\ \sin n\phi \\ 0 \end{pmatrix}; \quad (\text{A.2})$$

$$\hat{\phi}_2 = \psi_2(r) U_n(\phi) U_1(\Theta) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \psi_2(r) \begin{pmatrix} \cos(n\phi + \theta) \\ \sin(n\phi + \theta) \\ 0 \end{pmatrix}. \quad (\text{A.3})$$

The equations of motion for ϕ_1 are:

$$-\Delta\phi_1^1 - 2gA_\mu^3\partial_\mu\phi_1^2 - g^2(A_\mu^3)^2\phi_1^1 + 2\lambda_1(\phi_1^a \cdot \phi_1^a - F_1^2)\phi_1^1 + 2\kappa(\phi_1^a \cdot \phi_2^a - G^2)\phi_2^1 = 0, \quad (\text{A.4})$$

$$-\Delta\phi_1^2 + 2gA_\mu^3\partial_\mu\phi_1^1 - g^2(A_\mu^3)^2\phi_1^2 + 2\lambda_1(\phi_1^a \cdot \phi_1^a - F_1^2)\phi_2^2 + 2\kappa(\phi_1^a \cdot \phi_2^a - G^2)\phi_2^2 = 0, \quad (\text{A.5})$$

and analogous ones with $\Delta\phi_2^a$. Combining these one finds

$$\begin{aligned} & -\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right](\psi_1e^{in\phi})+2g iA_\phi^3\frac{1}{r}\partial_\phi(\psi_1e^{in\phi})+g^2(A_\phi^3)^2(\psi_1e^{in\phi}) \\ & +2\lambda_1(\phi_1^a\cdot\phi_1^a-F_1^2)(\psi_1e^{in\phi})+2\kappa(\phi_1^a\cdot\phi_2^a-G^2)(\psi_2e^{in\phi+i\theta})=0, \end{aligned} \quad (\text{A.6})$$

We now choose

$$\theta=\frac{\pi}{2}; \quad G=0 \quad (\text{A.7})$$

so that

$$-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\psi_1+\left(\frac{n}{r}-gA_\phi^3\right)^2\psi_1+2\lambda_1(\psi_1^2-F_1^2)\psi_1=0, \quad (\text{A.8})$$

The equation for ϕ_2 takes the same form, with $\{\psi_1, \lambda_1, F_1\} \rightarrow \{\psi_2, \lambda_2, F_2\}$. These equations have the same form as in the $U(1)$ theory [11]. The equation for the gauge field is:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}A_\phi^3-\frac{A_\phi^3}{r^2}+(\frac{gn}{r}-g^2A_\phi^3)(\psi_1^2+\psi_2^2)=0. \quad (\text{A.9})$$

To simplify further, let us take $\lambda_1=\lambda_2$, $F_1=F_2$, so $\psi_1=\psi_2\equiv\psi(r)$, and one has coupled equations

$$-\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\psi+\frac{(n-gA(r))^2}{r^2}\psi+2\lambda(\psi^2-F^2)\psi=0, \quad (\text{A.10})$$

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr}A(r)+2g\frac{n-gA(r)}{r}\psi^2=0, \quad (\text{A.11})$$

where we set $A_\phi^3(r)\equiv\frac{A(r)}{r}$. These can be solved easily numerically. If the relation

$$2\lambda=g^2 \quad (\text{A.12})$$

is satisfied, the above equations reduce further to the linear ones,

$$\frac{d}{dr}\psi=\frac{n-gA(r)}{r}\psi; \quad \frac{1}{r}\frac{d}{dr}A(r)=-g(\psi^2-F^2). \quad (\text{A.13})$$

as can be easily verified. One can further restrict oneself to the case of the minimum vortex with $n=1$, since $n=2$ vortices can be gauged away (see Appendix B). The profile of this vortex looks very similar to the $U(1)$ vortex of Nielsen-Olesen.

Appendix B: Gauge transformation which unwinds the $SO(3) = SU(2)/\mathbb{Z}_2$ “vortex” with flux $n=2$

The fact that the “vortex” of winding two does not represent a true vortex, follows from the group property, $\Pi_1(SO(3)) = \mathbb{Z}_2$. However, in this case it is easy to construct

explicitly, borrowing the idea from [18], a gauge transformation to “unwind” the apparent vortex-like configuration Eq.(2.5) with $n = 2$.

Define

$$U_n(\phi) = \begin{pmatrix} \cos n\phi & -\sin n\phi & 0 \\ \sin n\phi & \cos n\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{B.1})$$

$$\xi = \begin{pmatrix} \cos(\frac{\pi}{2+2r}) \cos^2 \phi + \sin^2 \phi & \{-1 + \cos(\frac{\pi}{2+2r})\} \cos \phi \sin \phi & -\cos \phi \sin(\frac{\pi}{2+2r}) \\ \{-1 + \cos(\frac{\pi}{2+2r})\} \cos \phi \sin \phi & \cos^2 \phi + \cos(\frac{\pi}{2+2r}) \sin^2 \phi & -\sin(\frac{\pi}{2+2r}) \sin \phi \\ \cos \phi \sin(\frac{\pi}{2+2r}) & \sin(\frac{\pi}{2+2r}) \sin \phi & \cos(\frac{\pi}{2+2r}) \end{pmatrix}, \quad (\text{B.2})$$

and

$$\eta = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}, \quad (\text{B.3})$$

where

$$\begin{aligned} a &= -\cos \phi \sin(\frac{\pi}{1+r}) \sin(\frac{\pi}{2+2r}) + \cos(\frac{\pi}{1+r}) \left\{ \cos(\frac{\pi}{2+2r}) \cos^2 \phi + \sin^2 \phi \right\}, \\ b &= \left\{ -1 + \cos(\frac{\pi}{2+2r}) \right\} \cos \phi \sin \phi, \\ c &= -\cos(\frac{\pi}{1+r}) \cos \phi \sin(\frac{\pi}{2+2r}) - \sin(\frac{\pi}{1+r}) \left\{ \cos(\frac{\pi}{2+2r}) \cos^2 \phi + \sin^2 \phi \right\}, \\ d &= -\left\{ \sin(\frac{\pi}{1+r}) \sin(\frac{\pi}{2+2r}) + 2 \cos(\frac{\pi}{1+r}) \cos \phi \sin^2(\frac{\pi}{4+4r}) \right\} \sin \phi, \\ e &= \cos^2 \phi + \cos(\frac{\pi}{2+2r}) \sin^2 \phi, \\ f &= -\cos(\frac{\pi}{1+r}) \sin(\frac{\pi}{2+2r}) \sin \phi + \sin(\frac{\pi}{1+r}) \sin^2(\frac{\pi}{4+4r}) \sin(2\phi), \\ g &= \cos(\frac{\pi}{2+2r}) \sin(\frac{\pi}{1+r}) + \cos(\frac{\pi}{1+r}) \cos \phi \sin(\frac{\pi}{2+2r}), \\ h &= \sin(\frac{\pi}{2+2r}) \sin \phi, \\ k &= \cos(\frac{\pi}{1+r}) \cos(\frac{\pi}{2+2r}) - \cos \phi \sin(\frac{\pi}{1+r}) \sin(\frac{\pi}{2+2r}), \end{aligned} \quad (\text{B.4})$$

and consider a gauge transformation,

$$\hat{\phi}_A \rightarrow U_{global}(r, \phi) \hat{\phi}_A; \quad A_i \rightarrow U_{global}(r, \phi) (A_i + \frac{1}{g} \partial_i) U_{global}^T(r, \phi), \quad (\text{B.5})$$

with

$$U_{global}(r, \phi) \equiv \eta(\phi, r) \cdot U_2(-\phi) \cdot \xi(\phi, r). \quad (\text{B.6})$$

It can be explicitly checked that $U_{global}(r, \phi)$ is regular everywhere and that

$$U_{global}(r, \phi) \xrightarrow{r \rightarrow \infty} U_2(-\phi); \quad U_{global}(r, \phi) \xrightarrow{r \rightarrow 0} \mathbf{1}, \quad (\text{B.7})$$

so that the “flux” disappears in the new gauge.